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Generalized Boltzmann Equation in a Manifestly Covariant Relativistic Statistical Mechanics

L. Burakovsky* and L.P. Horwitz†

School of Physics and Astronomy
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel-Aviv University, Tel-Aviv 69978, Israel

Abstract

We consider the relativistic statistical mechanics of an ensemble of N events with motion in space-time parametrized by an invariant "historical time" τ. We generalize the approach of Yang and Yao, based on the Wigner distribution functions and the Bogoliubov hypotheses, to find the approximate dynamical equation for the kinetic state of any nonequilibrium system to the relativistic case, and obtain a manifestly covariant Boltzmann-type equation which is a relativistic generalization of the Boltzmann-Uehling-Uhlenbeck (BUU) equation for indistinguishable particles. This equation is then used to prove the H-theorem for evolution in τ. In the equilibrium limit, the covariant forms of the standard statistical mechanical distributions are obtained. We introduce two-body interactions by means of the direct action potential $V(q)$, where q is an invariant distance in the Minkowski space-time. The two-body correlations are taken to have the support in a relative $O(2,1)$-invariant subregion of the full spacelike region. The expressions for the energy density and pressure are

*Bitnet: BURAKOV@TAUNIVM.TAU.AC.IL
†Bitnet: HORWITZ@TAUNIVM.TAU.AC.IL. Also at Department of Physics, Bar-Ilan University, Ramat-Gan, Israel

obtained and shown to have the same forms (in terms of an invariant distance parameter) as those of the nonrelativistic theory and to provide the correct nonrelativistic limit.

Key words: special relativity, relativistic Boltzmann equation, relativistic Maxwell-Boltzmann/Bose-Einstein/Fermi-Dirac, mass distribution

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1 Introduction

This paper continues a series of works on relativistic kinetic theory of an N-body system [1]–[7] within the framework of a manifestly covariant mechanics [8], both for the classical theory and the corresponding relativistic quantum theory. In this framework, for the classical case, the covariant dynamical evolution of a system of N particles is governed by equations of motion that are of the form of Hamilton equations for the motion of N *events* which generate the particle space-time trajectories (world lines). These events are considered as the fundamental dynamical objects of the theory and characterized by their positions $q^\mu = (ct, \mathbf{q})$ and energy-momenta $p^\mu = (E/c, \mathbf{p})$ in an $8N$-dimensional phase space. The motion is parametrized by a continuous Poincaré-invariant parameter τ [8] called the "historical time". For the quantum case, the covariant dynamical evolution of N particles is governed by a generalized Schrödinger equation for the wave function $\psi_\tau(q_1, q_2, ..., q_N) \in L^2(R^{4N})$, with measure $dq_1 dq_2 \cdots dq_N \equiv d^{4N}q$, describing the distribution of events $q_i \equiv q_i^\mu$, $\mu = 0, 1, 2, 3$; $i = 1, 2, \ldots, N$. The collection of events (called "concatenation" [9]) along each world line corresponds to a *particle* in the usual sense; e.g., the Maxwell conserved current is an integral over the history of the charged event [10]. Hence the evolution of the state of the N-event system describes *a posteriori* the history in space and time of an N-particle system.

The evolution of the system is assumed to be governed by Hamiltonian-type equations with a Lorentz-invariant scalar function, the relativistic dynamical function of the variables (q_i, p_i) specifying the state of each particle i. In the simplest case of a free particle, for which the world line is generated by a free event, the relativistic dynamical function (generalized Hamiltonian) is

$$K_0 = \frac{p^\mu p_\mu}{2M},$$

where we use the metric $g^{\mu\nu} = (-, +, +, +)$, and M is a given fixed parameter (an intrinsic property of the event), with the dimension of mass.

The Hamilton equations

$$\frac{dq^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu}, \quad \frac{dp^\mu}{d\tau} = -\frac{\partial K}{\partial q_\mu}$$

yield, in this case,

$$\frac{dq^\mu}{d\tau} = \frac{p^\mu}{M}, \quad \frac{dp^\mu}{d\tau} = 0.$$

Eliminating $d\tau$, one finds

$$\frac{d\mathbf{q}}{dt} = \frac{\mathbf{p}}{E}c^2,$$

as required for the motion of a free relativistic particle. It then follows that, for a free motion, the proper time interval squared, divided by $d\tau^2$, is

$$\frac{dq^\mu}{d\tau}\frac{dq_\mu}{d\tau} = \frac{p^\mu p_\mu}{M^2}.$$

For

$$K_0 = -\frac{M}{2},$$

corresponding to the "mass-shell" value

$$p^\mu p_\mu = -M^2 c^2,$$

it follows that

$$c^2 dt^2 - d\mathbf{q}^2 = c^2 d\tau^2.$$

In the more general case in which

$$K = K_0 + V,$$

where V is, for example, a function of q, $p^2 \equiv p^\mu p_\mu$ may vary from point to point along the trajectory. Hence, in general, the proper time interval does *not* correspond to $d\tau$.

For a system of N interacting events (and hence, particles) one takes [8]

$$K = \sum_i \frac{p_i^\mu p_{i\mu}}{2M} + V(q_1, q_2, \dots, q_N), \tag{1.1}$$

where all of the events are put, for simplicity, to have equal mass parameters, and we write q_i, for brevity, for the four-vector. The Hamilton equations are

$$\frac{dq_i^\mu}{d\tau} = \frac{\partial K}{\partial p_{i\mu}} = \frac{p_i^\mu}{M},$$

$$\frac{dp_i^\mu}{d\tau} = -\frac{\partial K}{\partial q_{i\mu}} = -\frac{\partial V}{\partial q_{i\mu}}. \tag{1.2}$$

These equations are precisely of the same form as those of nonrelativistic Hamilton point mechanics, but in a space of $8N$ dimensions instead of $6N$. The fundamental

theorems of mechanics, such as the Liouville theorem [2], the theory of canonical transformations and Hamilton-Jacobi theory, follow in the same way, with the manifold of space-time replacing that of space, and energy-momentum replacing the momentum. It is fundamental to this structure that there is a single universal evolution parameter τ which plays the role of the Galilean time.

In the quantum theory, the generalized Schrödinger equation

$$i\hbar\frac{\partial}{\partial\tau}\psi_\tau(q_1, q_2, \ldots, q_N) = K\psi_\tau(q_1, q_2, \ldots, q_N), \tag{1.3}$$

with, for example, a K of the form (1.1), describes the evolution of the N-body wave function $\psi_\tau(q_1, q_2, \ldots, q_N)$. To illustrate the meaning of this wave function, consider the case of a single free event. In this case, (1.3) has the formal solution

$$\psi_\tau(q) = (e^{-iK_0\tau/\hbar}\psi_0)(q)$$

for the evolution of the free wave packet. Let us represent $\psi_\tau(q)$ by its Fourier transform, in the energy-momentum space:

$$\psi_\tau(q) = \frac{1}{(2\pi\hbar)^2}\int d^4p\; e^{-ip^2\tau/2M\hbar}e^{ip\cdot q/\hbar}\psi_0(p),$$

where $p^2 \equiv p^\mu p_\mu,\ p\cdot q \equiv p^\mu q_\mu$, and $\psi_0(p)$ corresponds to the initial state. Applying the Ehrenfest arguments of stationary phase to obtain the principal contribution to $\psi_\tau(q)$ for a wave packet centered at p_c^μ, we find

$$q_c^\mu = \frac{p_c^\mu}{M}\tau,$$

consistent with the classical equations (1.2). Therefore, the central peak of the wave packet moves along the classical trajectory of an event, i.e., the classical world line.

The wave functions have a local interpretation, i.e., $|\psi_\tau(q)|^2 d^4q$ is the probability to find an event at the space-time point q^μ in space-time volume d^4q. Localization in space, as well as in time, can be shown by applying arguments given in ref. [11].

Horwitz, Schieve and Piron [1] have constructed equilibrium classical and quantum Gibbs ensembles. They found that the grand partition function in the rest frame of the system is given by

$$\ln Z(\beta, V, \mu, \mu_K) = e^{\beta\mu}\int\frac{d^4pd^4q}{(2\pi)^4}e^{-\beta E}e^{\beta\mu_K\frac{m^2}{2M}},\quad \beta = \frac{1}{k_BT}. \tag{1.4}$$

In addition to the usual chemical potential μ in the grand canonical ensemble, there is a new potential μ_K corresponding to the mass degree of freedom of relativistic systems.

Horwitz, Shashoua and Schieve [2] have shown that in the framework of the manifestly covariant mechanics which we discuss here, covariant Weyl transforms exist

for observables, and therefore covariant relativistic Wigner functions [12] can be constructed. In this way they derived a manifestly covariant relativistic generalization of the BBGKY hierarchy for the s-particle relativistic Wigner functions. By approximating the effect of correlation of second and higher order by two-body collision terms (using the cross-sections defined in ref. [13]), as in the usual nonrelativistic Boltzmann theory, they obtained a manifestly covariant Boltzmann equation (for non-identical events). This equation was used to prove the H-theorem for evolution in τ. In the equilibrium limit, a covariant form of the Maxwell-Boltzmann distribution,

$$f^{(0)}(q,p) = e^{A(q)(p-p_c)^2}, \tag{1.5}$$

was obtained. Since this distribution is the distribution of the 4-momenta of the events, $m^2 = -p^2 = -p^\mu p_\mu$ is a random variable in a relativistic ensemble. In order to obtain a simple analytic result the authors restricted themselves to a narrow mass shell $p^2 = -m^2 \cong -M^2$. The results obtained in this approximation are in agreement with the well-known results of Synge [14] for an on-shell relativistic kinetic theory.

In ref. [3] the equilibrium Maxwell-Boltzmann distribution (1.5) was considered for the whole range of m, to obtain the corresponding equilibrium relativistic *mass* distribution. Its low-temperature and nonrelativistic limits were investigated and shown to yield results in agreement with nonrelativistic statistical mechanics [5].

In the present paper we study the case of indistinguishable events. In contrast to the approach of Horwitz, Shashoua and Schieve [2], we choose another approach initiated by Yang and Yao [15] in the nonrelativistic case, which is based on the Wigner distribution functions and the Bogoliubov hypotheses to find approximate dynamical equation for the kinetic state of any nonequilibrium system [16]. Kinetic equation that we obtain, which represents a relativistic generalization of the Boltzmann-Uehling-Uhlenbeck (BUU) equation [17] for indistinguishable particles, and can be easily generalized to include the non-identical case as well. The generalized Boltzmann equation obtained in this way is then used to prove the H-theorem for evolution in τ. In the equilibrium limit, the covariant forms of the Bose-Einstein/Fermi-Dirac/Maxwell-Boltzmann distributions are obtained, which, as considered for the whole range of m, provide the corresponding equilibrium relativistic mass distributions. The relativistic mass distributions are studied in the identical particle case in [4], and their possible consequences for high energy physics and cosmology are considered, respectively, in [7] and [18].

We introduce two-body interactions by taking the support of mutual correlations for any two events to be in a relative $O(2,1)$-invariant subregion of the full spacelike region, as done in the solution of the two-body bound state problem [19, 20], and for the extraction of the partial wave expansion from the relativistic scattering amplitude [21]. We then calculate the expressions for the energy density and pressure of an interacting gas, and show that they have the same form (in terms of an invariant distance parameter) as those of the nonrelativistic theory and provide the correct nonrelativistic limit.

2 Relativistic N-body system

The evolution in τ of an N-body system is determined by the Liouville-von Neumann equation for the N-body density matrix ρ (we use the system of units in which $\hbar = c = k_B = 1$, unless other units are specified):

$$i\frac{\partial \rho}{\partial \tau} = [K, \rho], \tag{2.1}$$

where K is the total N-body Hamiltonian, here taken to be

$$K = \sum_{i=1}^{N} K_i^{(0)} + \sum_{1=i<j}^{N} V_{i,j}, \tag{2.2}$$

where

$$K_i^{(0)} = \frac{p_i^{\mu} p_{i\mu}}{2M}$$

and

$$V_{i,j} = V\left(q_i - q_j\right), \quad q_i - q_j \equiv \sqrt{(q_i^{\mu} - q_j^{\mu})(q_{i\mu} - q_{j\mu})}$$

is a two-body interaction potential. In order to obtain the BBGKY hierarchy, one introduces the (n)-body density matrices, as follows:

$$\rho_{1,2,\ldots,n}^{(n)} = \frac{N!}{(N-n)!} Tr_{(n+1,\ldots,N)}\rho, \tag{2.3}$$

$$Tr_{(1,2,\ldots,n)}\rho_{1,2,\ldots,n}^{(n)} = \frac{N!}{(N-n)!}, \tag{2.4}$$

and, by taking the appropriate traces in Eq. (2.1), obtains [22]

$$\begin{aligned} i\frac{\partial \rho_{1,2,\ldots,n}^{(n)}}{\partial \tau} &= \sum_{i=1}^{n} [K_i^{(0)}, \rho_{1,2,\ldots,n}^{(n)}] + \sum_{1=i<j}^{n} [V_{i,j}, \rho_{1,2,\ldots,n}^{(n)}] \\ &+ Tr_{(n+1)} \sum_{i=1}^{n} [V_{i,n+1}, \rho_{1,2,\ldots,n+1}^{(n+1)}]. \end{aligned} \tag{2.5}$$

This set of equations is equivalent to (2.1).

In what follows, we shall use the simplified notation: $\rho_i \equiv \rho_i^{(1)}$, $\rho_{i,j} \equiv \rho_{i,j}^{(2)}$, etc., so that the latter equation can be rewritten as

$$i\frac{\partial \rho_n}{\partial \tau} = \sum_{i=1}^{n} [K_i^{(0)}, \rho_n] + \sum_{1=i<j}^{n} [V_{i,j}, \rho_n] + Tr_{(n+1)} \sum_{i=1}^{n} [V_{i,n+1}, \rho_{n+1}]. \tag{2.6}$$

It is convenient to introduce directly the symmetry requirements on the function ρ_n by means of

$$\rho_n = S_n F_n, \tag{2.7}$$

where S_n is a symmetrization/antisymmetrization operator defined by

$$S_n = \prod_{i=2}^{n} \left(1 \pm \sum_{j=1}^{i-1} P_{i,j} \right). \tag{2.8}$$

Here $P_{i,j}$ denotes the permutation operator. Since S_n satisfies the relation

$$S_{n+1} = S_n \left(1 \pm \sum_{i=1}^{n} P_{i,n+1} \right) \tag{2.9}$$

and commutes with the operators K_i and $V_{i,j}$, one can substitute (2.7) into (2.6) and obtain the equation

$$\begin{aligned} i\frac{\partial F_n}{\partial \tau} \;\; = \;\; & \sum_{i=1}^{n} [K_i^{(0)}, F_n] \; + \sum_{1=i<j}^{n} [V_{i,j}, F_n] \; + \; Tr_{(n+1)} \sum_{i=1}^{n} [V_{i,n+1}, F_{n+1}] \\ & \pm \; Tr_{(n+1)} \sum_{i=1}^{n} [V_{i,n+1}, \sum_{i=1}^{n} P_{i,n+1} F_{n+1}]. \end{aligned} \tag{2.10}$$

Now we introduce the Wigner distribution functions [12],

$$f_s(q_s, p_s, \tau) = \frac{1}{(2\pi)^{4s}} \int dr_s \; F_s(q_s^{'}, q_s^{''}, \tau) e^{ip_s \cdot r_s}, \tag{2.11}$$

$$F_s(q_s^{'}, q_s^{''}, \tau) = \int dp_s \; f_s(q_s, p_s, \tau) e^{-ip_s \cdot r_s}, \tag{2.12}$$

where

$$q_s^{'} = q_s - \frac{1}{2} r_s, \quad q_s^{''} = q_s + \frac{1}{2} r_s,$$

and $q_s \equiv (q_1, q_2, \ldots, q_s)$, $p_s \equiv (p_1, p_2, \ldots, p_s)$, $p_s \cdot r_s \equiv \sum_{i=1}^{s} p_i^{\mu} p_{i\mu}$, $dr_s \equiv dr_1 dr_2 \cdots dr_s$. One may substitute (2.12) into (2.10) and obtain the quantum BBGKY hierarchy of the Wigner distribution functions $f_s = f_s(x_s, \tau)$, $x_s = (q_s, p_s)$, as

$$\begin{aligned} \frac{\partial f_s}{\partial \tau} \;\; & + \;\; \sum_{j=1}^{s} \frac{p_j}{M} \frac{\partial f_s}{\partial q_j} \;\; + \;\; i \sum_{j<k}^{s} \left(e^{i\theta_{j,k}/2} - e^{-i\theta_{j,k}/2} \right) f_s \\ & + \;\; i \sum_{j=1}^{s} \int dx_{s+1} \;\; \left(e^{i\theta_{j,s+1}/2} - e^{-i\theta_{j,s+1}/2} \right) f_{s+1} \\ & \pm \;\; i \sum_{j=1}^{s} \int dx_{s+1} \;\; \left(e^{i\theta_{j,s+1}/2} - e^{-i\theta_{j,s+1}/2} \right) P_{j,s+1} f_{s+1} \; = \; 0. \end{aligned} \tag{2.13}$$

Here $dx_s \equiv dq_s dp_s = d^4q_1 \cdots d^4q_s d^4p_1 \cdots d^4p_s$, and the operators θ_{ij} and $\theta_{j,s+1}$ are represented as follows,

$$\theta_{ij} = \frac{\partial V_{ij}}{\partial q_i} \left(\frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_j} \right), \quad \theta_{j,s+1} = \frac{\partial V_{ij}}{\partial q_i} \frac{\partial}{\partial p_j}. \tag{2.14}$$

For $s = 1$ and 2, one finds

$$\begin{aligned} \frac{\partial f_1}{\partial \tau} &+ \frac{p_1}{M}\frac{\partial f_1}{\partial q_1} + i\int dx_2 \left(e^{i\theta_{1,2}/2} - e^{-i\theta_{1,2}/2}\right) f_2 \\ &\pm i\int dx_2 \left(e^{i\theta_{1,2}/2} - e^{-i\theta_{1,2}/2}\right) P_{1,2} f_2 = 0, \end{aligned} \tag{2.15}$$

$$\begin{aligned} \frac{\partial f_2}{\partial \tau} &+ \sum_{j=1}^{2} \frac{p_j}{M}\frac{\partial f_2}{\partial q_j} + i\left(e^{i\theta_{1,2}/2} - e^{-i\theta_{1,2}/2}\right) f_2 \\ &+ i\sum_{j=1}^{2}\int dx_3 \left(e^{i\theta_{j,3}/2} - e^{-i\theta_{j,3}/2}\right) f_3 \\ &\pm i\sum_{j=1}^{2}\int dx_3 \left(e^{i\theta_{j,3}/2} - e^{-i\theta_{j,3}/2}\right) P_{j,3} f_3 = 0. \end{aligned} \tag{2.16}$$

Equations (2.15) and (2.16) are exact. Since f_2 depends on f_3, accurate solution of the hierarchy is impossible. One has, therefore, to apply some approximated approach. One of such approaches is the Bogoliubov one [16], which we shall apply in the present consideration.

According to the Bogoliubov hypotheses [16],

1) It is possibe to find a kinetic state of any non-equilibrium system, provided that the average interval between two subsequent collisions is much longer than the duration of the collision. In this kinetic state,

$$f_s(x_1, \ldots, x_s; \tau) = f_s(x_1, \ldots, x_s | f_1), \tag{2.17}$$

$$\frac{\partial f_1}{\partial \tau} = A(x|f_1). \tag{2.18}$$

2) There are no correlations in the initial state of a system. One can introduce the displacement operator,

$$\mathcal{P}^s_\tau f_s(x_1^0, \ldots, x_s^0) = f_s(x_1, \ldots, x_s), \tag{2.19}$$

where $x_1^0, \ldots, x_s^0$ are the values of each x at $\tau = 0$, and $x_1, \ldots, x_s$ are their values at τ. The non-correlative condition at the initial state implies

$$\mathcal{P}^s_{-\tau}\left[f_s(x_1, \ldots, x_s) - \prod_{1\le j\le s} f_1(x_j)\right] \to 0. \tag{2.20}$$

Starting from the Bogoliubov hypotheses, it is possible to derive a kinetic equation.

Although the invariant interaction potential has infinite support in space-time, since it depends on $(\mathbf{q}_1 - \mathbf{q}_2)^2 - c^2(t_1 - t_2)^2$, its long-range part is necessary close to the light cone. It has been shown [23], that wave operators exist in scattering theory if the support of the wave function does not extend to zero mass. The space-time

volume v of the effective interaction is therefore bounded. We shall assume here that it may be taken to be small, as in the first hypothesis of Bogoliubov. One can, therefore, write

$$\frac{\partial f_1}{\partial \tau} = A^0(x|f_1) + vA^1(x|f_1) = \ldots, \tag{2.21}$$

$$f_s = f_s^0 + vf_s^1 + v^2 f_s^2 + \ldots. \tag{2.22}$$

In the first-order approximation, one sets

$$f_2 \cong f_2^0 \cong f_1(1)f_1(2) \tag{2.23}$$

(henceforth we use the notation $1 \equiv (x_1; \tau)$, $2 \equiv (x_2; \tau)$, etc.) and finds from (2.15)

$$\begin{aligned} \frac{\partial f_1(1)}{\partial \tau} \quad + \quad \frac{p_1}{M}\frac{\partial f_1(1)}{\partial q_1} \quad + \quad i\int dx_2 \, \left(e^{i\theta_{1,2}/2} - e^{-i\theta_{1,2}/2}\right) f_1(1)f_1(2) \\ \pm \quad i\int dx_2 \, \left(e^{i\theta_{1,2}/2} - e^{-i\theta_{1,2}/2}\right) P_{1,2}f_1(1)f_1(2) \quad = \quad 0. \end{aligned} \tag{2.24}$$

This self-consistent equation is a relativistic generalization of the quantum Vlasov equation [15].

In the second-order approximation, one writes a formal solution,

$$f_s(x_1, \ldots, x_s|f_1) = \sum_{i<j\leq s} g(x_i, x_j) \prod_{k\neq i\neq j} f_1(k), \tag{2.25}$$

where

$$g(x_i, x_j) = f_2^1(x_i, x_j|f_1) \tag{2.26}$$

is a two-body correlation function, whose boundary condition is

$$\lim_{\tau\to\infty} \mathcal{P}_{-\tau}^{(2)} g(x_i, x_j) = 0. \tag{2.27}$$

Eq. (2.25) means that s-body effects are correlated by two-body effects. One can write

$$\begin{aligned} \frac{\partial f_2}{\partial \tau} \quad = \quad \frac{\partial f_2}{\partial f_1}\frac{\partial f_1}{\partial \tau} \quad &\approx \quad \left(\frac{\partial f_2^0}{\partial f_1} + v\frac{\partial f_2^1}{\partial f_1}\right)\left[A^0(x|f_1) + vA^1(x|f_1)\right] \\ &\approx \quad D_0 f_2^0 \; + \; v\left[D_0 \, g(x_1, x_2) + D_1 f_2^0\right], \end{aligned} \tag{2.28}$$

where

$$D_0 \equiv A^0 \frac{\partial}{\partial f_1}, \quad D_1 \equiv A^1 \frac{\partial}{\partial f_1}.$$

One now uses Eqs. (2.16),(2.25) and obtains

$$
\begin{aligned}
D_0\, g(x_1,x_2) \;&+\; \sum_{j=1}^{2}\frac{p_j}{M}\frac{\partial}{\partial q_j}g(x_1,x_2) \;+\; i\sum_{j=1}^{2}\left(e^{i\eta_j/2}-e^{-i\eta_j/2}\right)g(x_1,x_2) \\
&= \;-\; i\left(e^{i\theta'_{1,2}/2}-e^{-i\theta'_{1,2}/2}\right)f_1(1)f_1(2) \;-\; i\int dx_3\left(e^{i\theta'_{1,3}/2}-e^{-i\theta'_{1,3}/2}\right) \\
&\quad \times g(x_2,x_3)f_1(1) \;-\; i\int dx_3\left(e^{i\theta'_{2,3}/2}-e^{-i\theta'_{2,3}/2}\right)f_1(2)g(x_1,x_3) \\
&\quad \mp\; i\int dx_3\left(e^{i\theta'_{1,3}/2}-e^{-i\theta'_{1,3}/2}\right)f_1(1)f_1(2)f_1(3) \\
&\quad \mp\; i\int dx_3\left(e^{i\theta'_{2,3}/2}-e^{-i\theta'_{2,3}/2}\right)f_1(1)f_1(2)f_1(3). \qquad (2.29)
\end{aligned}
$$

Once $g(x_1,x_2)$ is known, one can obtain the two-order-approximated equation for f_1 :

$$
\begin{aligned}
\frac{\partial f_1(1)}{\partial\tau} \;&+\; \frac{p_1}{M}\frac{\partial f_1(1)}{\partial q_1} \;+\; i\left(e^{i\eta_1/2}-e^{-i\eta_1/2}\right)f_1(1) \;+\; i\int dx_2\,\left(e^{i\theta'_{1,2}/2}-e^{-i\theta'_{1,2}/2}\right) \\
&\times g(x_1,x_2) \;\pm\; i\int dx_2\,\left(e^{i\theta'_{1,2}/2}-e^{-i\theta'_{1,2}/2}\right)f_1(1)f_1(2) \;=\; 0. \qquad (2.30)
\end{aligned}
$$

Here

$$\theta'_{1,2}=\frac{1}{v}\theta_{1,2},\;\; \theta'_{1,3}=\frac{1}{v}\theta_{1,3},\;\; \eta_1=\frac{\partial U_1}{\partial q_1}\frac{\partial}{\partial p_1},$$

and

$$U_1(q_1,\tau)=\frac{1}{v}\int dx_2\; f_1(2)V(q_1-q_2) \qquad (2.31)$$

is the mean-field potential. In general, it is very difficult to obtain simultaneously solutions of Eqs. (2.29) and (2.30). In the following section we show how Eq. (2.29) can be solved for a quasihomogeneous system.

2.1 Quasihomogeneous system

The condition on a quasihomogeneous system is

$$g(x_1,x_2)=g(q_1-q_2,p_1,p_2)\equiv g(q,p_1,p_2), \qquad (2.32)$$

i.e., the correlation function depends only on the relative coordinates. In this case, one obtains a formal solution for $g(q,p_1,p_2)$ by means of the displacement techniques, as follows:

$$
\begin{aligned}
g(q,p_1,p_2) \;&=\; \int_0^\infty d\tau\;\left[i\left\{\left(e^{\frac{i}{2}\frac{\partial}{\partial q}(\frac{\partial}{\partial p_1}-\frac{\partial}{\partial p_2})}-e^{-\frac{i}{2}\frac{\partial}{\partial q}(\frac{\partial}{\partial p_1}-\frac{\partial}{\partial p_2})}\right)V\left(q-\frac{p_1-p_2}{M}\tau\right)\right\}\right. \\
&\quad \times f_1(1)f_1(2) \;+\; i\int dq'dp_3\;\left(e^{\frac{i}{2}\frac{\partial}{\partial q}\frac{\partial}{\partial p_1}}-e^{-\frac{i}{2}\frac{\partial}{\partial q}\frac{\partial}{\partial p_1}}\right)V\left(q-q'\right.
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{p_1 - p_2}{M}\tau\Big) \times \Big(g(q', p_2, p_3) f_1(1) \ \pm \ f_1(1) f_1(2) f_1(3)\Big) \\
& \pm i \int dq' dp_3 \ \left(e^{\frac{i}{2}\frac{\partial}{\partial q}\frac{\partial}{\partial p_2}} - e^{-\frac{i}{2}\frac{\partial}{\partial q}\frac{\partial}{\partial p_2}}\right) V\left(q - q' - \frac{p_1 - p_2}{M}\tau\right) \\
& \times \Big(g(q', p_1, p_3) f_1(2) \ \pm \ f_1(1) f_1(2) f_1(3)\Big)\Big] \\
= \ & i \int d\tau \ \Big[\Big(e^{i\theta'_{1,2}/2} - e^{-i\theta'_{1,2}/2}\Big) f_1(1) f_1(2) \\
& + \int dx_3 \ \Big(e^{i\theta'_{1,3}/2} - e^{-i\theta'_{1,3}/2}\Big) \times \Big(g(x_2, x_3) f_1(1) \ \pm \ f_1(1) f_1(2) f_1(3)\Big) \\
& \pm \int dx_3 \ \Big(e^{i\theta'_{2,3}/2} - e^{-i\theta'_{2,3}/2}\Big) \times \Big(g(x_1, x_3) f_1(2) \ \pm \ f_1(1) f_1(2) f_1(3)\Big)\Big].
\end{aligned}
\tag{2.33}
$$

In order to solve Eqs. (2.30) and (2.33), it is convenient to introduce the Fourier transform, as follows:

$$
\tilde{g}(k, p_1, p_2) = \int dq \ g(q, p_1, p_2) e^{-ik\cdot q}, \tag{2.34}
$$

$$
\tilde{V}(k) = \int dq \ V(q) e^{-ik\cdot q}. \tag{2.35}
$$

Substituting Eqs. (2.34),(2.35) into (2.30), one finds

$$
\frac{\partial f_1}{\partial \tau} + \frac{p_1}{M}\frac{\partial f_1}{\partial q_1} + F\frac{\partial f_1}{\partial p_1} = -\frac{i}{(2\pi)^4}\int dk \ \left(e^{\frac{k}{2}\frac{\partial}{\partial p_1}} - e^{-\frac{k}{2}\frac{\partial}{\partial p_1}}\right)\tilde{V}_{1,2}(k) h(k, p_1), \tag{2.36}
$$

where

$$
h(k, p_1) = \int dp_2 \ g(k, p_1, p_2), \tag{2.37}
$$

$$
F\frac{\partial f_1}{\partial p_1} = i\left(e^{i\eta_1/2} - e^{-i\eta_1/2}\right) f_1(1) \pm i\int dx_2 \ \left(e^{i\theta'_{1,2}} - e^{-i\theta'_{1,2}}\right) f_1(1) f_1(2). \tag{2.38}
$$

Making a Fourier transform of Eq. (2.33), one obtains, after some manipulations,

$$
\begin{aligned}
\mathrm{Im}\ h(k, p_1) \ \ = \ \ & \int dp_2 \ \frac{\pi \tilde{V}_{1,2}(k)}{k|1 \mp \tilde{V}_{2,3}\psi|^2}\left[f_1^+(1) f_1^-(2) - f_1^+(2) f_1^-(1)\right] \\
& \times \delta\left(k \cdot \frac{p_1 - p_2}{M}\right).
\end{aligned}
\tag{2.39}
$$

Here

$$
f^{\pm} = f\left(p \pm \frac{k}{2}\right)\left[1 \pm f\left(p \mp \frac{k}{2}\right)\right] \tag{2.40}
$$

(the second sign $\pm$ in (2.40) distinguishes between bosons and fermions),

$$
f\left(p \pm \frac{k}{2}\right) = e^{\pm\frac{k}{2}\frac{\partial}{\partial p}} f(p), \tag{2.41}
$$

and

$$\psi = \int_{-\infty}^{\infty} \frac{dp_3}{k \cdot (\frac{p_1 - p_2}{M}) - i\varepsilon} \left[f_1^+(3) - f_1^-(3) \right]. \tag{2.42}$$

Substituting (2.39) into (2.36), one finally obtains

$$\begin{aligned} \frac{\partial f_1(1)}{\partial \tau} \quad + \quad \frac{p_1}{M}\frac{\partial f_1(1)}{\partial q_1} \quad + \quad F\frac{\partial f_1(1)}{\partial p_1} \quad &= \quad \frac{\pi}{(2\pi)^4} \int dk \, \left(e^{\frac{k}{2}\frac{\partial}{\partial p_1}} - e^{-\frac{k}{2}\frac{\partial}{\partial p_1}} \right) \\ \times \int dp_2 \, \delta \left(k \cdot \frac{p_1 - p_2}{M} \right) &\frac{\tilde{V}_{1,2}^2(k)}{|1 \mp \tilde{V}_{2,3}\psi|^2} \left[f_1^+(1) f_1^-(2) - f_1^+(2) f_2^-(1) \right]. \end{aligned} \tag{2.43}$$

Equation (2.43) is the kinetic equation of a gas of indistinguishable particles in the quasihomogeneous case (the improved BUU equation [15]). It reduces to the usual BUU equation provided that the many-body effects are neglected and that the first-order approximation for the term F is taken:

$$\begin{aligned} \frac{\partial f_1(1)}{\partial \tau} \quad + \quad \frac{p_1}{M}\frac{\partial f_1(1)}{\partial q_1} \quad - \quad \frac{\partial U_1}{\partial q_1}\frac{\partial f_1(1)}{\partial p_1} \quad &= \quad \frac{\pi}{(2\pi)^{12}} \int dp_2 dp_1^{'} dp_2^{'} \, \delta^4(p_1 + p_2 - p_1^{'} - p_2^{'}) \\ \times \left| \langle p_1 p_2 | V_{1,2} | p_1^{'} p_2^{'} \rangle \right|^2 &\left\{ f_1(1^{'}) f_1(2^{'}) \left[1 \pm f_1(1) \right] \left[1 \pm f_1(2) \right] \right. \\ - \quad f_1(1) f_1(2) &\left. \left[1 \pm f_1(1^{'}) \right] \left[1 \pm f_1(2^{'}) \right] \right\}. \end{aligned} \tag{2.44}$$

In contrast to the usual Boltzmann and BUU equations which are applicable in the restriction on the system to be dilute, Eq. (2.44) includes the influence of many-body effects. Therefore, Eq. (2.44) provides an essential improvement for the systems that have a higher particle density or a larger force range of particle interaction; e.g., for strongly interacting matter, heavy-ion collisions, or a cold relativistic plasma.

Rewriting this equation in the form

$$\begin{aligned} \frac{\partial f_1(1)}{\partial \tau} \quad + \quad \frac{p_1}{M}\frac{\partial f_1(1)}{\partial q_1} \quad - \quad \frac{\partial U_1}{\partial q_1}\frac{\partial f_1(1)}{\partial p_1} \quad &= \quad \frac{\pi}{(2\pi)^{12}} \int dp_2 dp_1^{'} dp_2^{'} \, \delta^4(p_1 + p_2 - p_1^{'} - p_2^{'}) \\ \times \left| \langle p_1 p_2 | V_{1,2} | p_1^{'} p_2^{'} \rangle \right|^2 &\left\{ f_1(1^{'}) f_1(2^{'}) \left[1 + \sigma f_1(1) \right] \left[1 + \sigma f_1(2) \right] \right. \\ - \quad f_1(1) f_1(2) &\left. \left[1 + \sigma f_1(1^{'}) \right] \left[1 + \sigma f_1(2^{'}) \right] \right\}, \quad \sigma = \pm 1, \end{aligned} \tag{2.45}$$

one sees that it reduces to the usual Boltzmann equation for non-identicai particles for $\sigma = 0$. Thus, the three cases,

$$\sigma = \begin{cases} 1, & \text{Bose} - \text{Einstein}, \\ -1, & \text{Fermi} - \text{Dirac}, \\ 0, & \text{Maxwell} - \text{Boltzmann}, \end{cases}$$

can be treated by means of a unique equation, (2.45), which can be, therefore, called the generalized Boltzmann equation.

3 Boltzmann H-theorem

We now wish to establish the relativistic analogue to the Boltzmann H-theorem and to prove that the entropy of an ensemble of events, evolving without external disturbances, is nondecreasing as a function of τ.

The density of states in phase space associated with the distribution n has been found in [1],

$$\triangle\Gamma(\bar{n}) = \begin{cases} (\bar{n}+g-1)!/\bar{n}!(g-1)!, & \text{Bose} - \text{Einstein} \\ g!/\bar{n}!(g-\bar{n})!, & \text{Fermi} - \text{Dirac} \\ g^{\bar{n}}/\bar{n}! & \text{Maxwell} - \text{Boltzmann} \end{cases}$$

where g is a number of states in each elementary cell of energy-momentum space (degeneracy) and $\bar{n}$ is the average occupation number. Assuming no degeneracy ($g \rightarrow 1$) and using Stirling's approximation

$$\ln N! \approx N \ln N, \quad N >> 1,$$

we obtain for the density of entropy in phase space, s,

$$\begin{aligned} \frac{s}{k_B} \equiv \ln\triangle\Gamma(\bar{n}) &= \begin{cases} -\bar{n}\ln\bar{n} + (1+\bar{n})\ln(1+\bar{n}), & \text{Bose} - \text{Einstein} \\ -\bar{n}\ln\bar{n} - (1-\bar{n})\ln(1-\bar{n}), & \text{Fermi} - \text{Dirac} \\ -\bar{n}\ln\bar{n}, & \text{Maxwell} - \text{Boltzmann} \end{cases} \\ &= (\sigma+\bar{n})\ln(1+\sigma\bar{n}) - \bar{n}\ln\bar{n}, \quad \sigma = \pm 1, 0. \end{aligned} \tag{3.1}$$

Therefore, in the case we are considering, the entropy of the ensemble is defined by the functional

$$\frac{S(\tau)}{k_B} = \int dqdp\, \Big[(\sigma + f_1(q,p;\tau))\ln(1+\sigma f_1(q,p;\tau)) - f_1(q,p;\tau)\ln f_1(q,p;\tau)\Big]. \tag{3.2}$$

Then, taking the derivative of $S(\tau)/k_B$, using Eq. (2.45) and integration by parts of the space-time derivatives, we obtain, after some manipulations,

$$\begin{aligned} \frac{1}{k_B}\frac{dS}{d\tau} &= \frac{\pi}{4(2\pi)^{12}}\int dqdp_1dp_2dp_1^{'}dp_2^{'}\delta^4(p_1+p_2-p_1^{'}-p_2^{'})\Big|\langle p_1p_2|V_{1,2}|p_1^{'}p_2^{'}\rangle\Big|^2 \\ &\times f_1(q,p_1;\tau)f_1(q,p_2;\tau)f_1(q,p_1^{'};\tau)f_1(q,p_2^{'};\tau)\left[\left(\frac{1}{f_1(q,p_1;\tau)}+\sigma\right)\right. \\ &\times\left.\left(\frac{1}{f_1(q,p_2;\tau)}+\sigma\right) - \left(\frac{1}{f_1(q,p_1^{'};\tau)}+\sigma\right)\left(\frac{1}{f_1(q,p_2^{'};\tau)}+\sigma\right)\right] \\ &\times\left[\ln\left\{\left(\frac{1}{f_1(q,p_1;\tau)}+\sigma\right)\left(\frac{1}{f_1(q,p_2;\tau)}+\sigma\right)\right\}\right. \\ &- \left.\ln\left\{\left(\frac{1}{f_1(q,p_1^{'};\tau)}+\sigma\right)\left(\frac{1}{f_1(q,p_2^{'};\tau)}+\sigma\right)\right\}\right]. \end{aligned} \tag{3.3}$$

In the derivation of (3.3) the principle of microscopic irreversibility (e.g., detailed balance)

$$\left|\langle p_1 p_2|V_{1,2}|p_1' p_2'\rangle\right|^2 dp_1 dp_2 = \left|\langle p_1' p_2'|V_{1,2}|p_1 p_2\rangle\right|^2 dp_1' dp_2'$$

and the hermiticity condition

$$\left|\langle p_1 p_2|V_{1,2}|p_1' p_2'\rangle\right|^2 = \left|\langle p_1' p_2'|V_{1,2}|p_1 p_2\rangle\right|^2$$

were used. Since $\left|\langle p_1 p_2|V_{1,2}|p_1' p_2'\rangle\right|^2 \delta^4(p_1 + p_2 - p_1' - p_2') \geq 0$, and the remaining factor in the integrand is non-negative, we obtain

$$\frac{dS(\tau)}{d\tau} \geq 0, \tag{3.4}$$

the relativistic Boltzmann H-theorem.

This result implies that the entropy $S(\tau)$ is monotonically increasing as a function of τ, and hence the evolution of the system, as described by the generalized relativistic Boltzmann equation, is irreversible in τ, but *not necessarily in t.* In a smooth average sense, one can argue that the entropy must increase in t as well. The support of the distribution function in t is finite at each τ; as τ increases, this supprort moves up the t-axis, since the system as a whole moves with the free motion of the center of mass. The entropy, according to the H-theorem in τ, must therefore also be nondecreasing, in this coarse-grained sense, in t. In the nonrelativistic limit [24] $t \to \tau$, $S(t)$ takes on the usual nonrelativistic form, and the nonrelativistic H-theorem for evolution in t is recovered.

In the special case in which the ensemble consists of positive energy (or negative energy) states alone, a precise H-theorem can be proved for the Lyapunov function

$$\frac{\tilde{S}(t)}{k_B} = \int d^3q\, d\tau \int_{p^0>0} d^4p\, \frac{p^0}{M} \Big[(\sigma + f_1(q,p;\tau)) \ln\, (1 + \sigma f_1(q,p;\tau)) - f_1(q,p;\tau) \ln f_1(q,p;\tau)\Big],$$

by the application of the arguments contained in ref. [2].

3.1 Relativistic four-momentum distributions

As we have seen in the preceding section, the entropy (3.2) of a system of events increases, according to the generalized relativistic Boltzmann equation, monotonically in τ. It means that the momentum distribution function monotonically approaches its equilibrium value $f_1^{(0)}(q,p)$. The equilibrium limit is achieved when

$$\frac{dS(\tau)}{d\tau} = 0. \tag{3.5}$$

Since the integrand in (3.3) is definite, (3.5) requires that, for the equilibrium distribution $f_1^{(0)}(q,p)$,

$$\ln\left(\frac{1}{f_1^{(0)}(q,p_1)}+\sigma\right)+\ln\left(\frac{1}{f_1^{(0)}(q,p_2)}+\sigma\right)=\ln\left(\frac{1}{f_1^{(0)}(q,p_1^{'})}+\sigma\right)+\ln\left(\frac{1}{f_1^{(0)}(q,p_2^{'})}+\sigma\right); \tag{3.6}$$

this condition implies the vanishing of the collision term in the generalized relativistic Boltzmann equation (2.45).

Since p_1, p_2 and $p_1^{'},\ p_2^{'}$ are the initial and final four-momenta for any scattering process, the general solution of (3.6) is of the form

$$\ln\left(\frac{1}{f_1^{(0)}(q,p)}+\sigma\right)=\chi_1(q,p)+\chi_2(q,p)+\ldots, \tag{3.7}$$

where the χ_i exhaust all quantities for which

$$\chi_i(q,p_1)+\chi_i(q,p_2) \tag{3.8}$$

are conserved in collisions. The quantities conserved in the sense of (3.8) are the individual event four-momentum p^μ and mass squared $m^2\equiv -p^2$ (the latter is asymptotically conserved in the scattering process [13]), and a constant (the one-event "angular momentum" $M^{\mu\nu}=q^\mu p^\nu-q^\nu p^\mu$ also satisfies this requirement, but does not change the structure of the result). Hence, the most general form of $f_1^{(0)}$ is given by [2, 3]

$$\ln\left(\frac{1}{f_1^{(0)}(q,p)}+\sigma\right)=-A(p-p_c)^2-B,\quad A=A(q),\ B=B(q), \tag{3.9}$$

where $p^\mu p_{c\mu}$ is an arbitrary linear combination of the components p^μ, so that

$$f_1^{(0)}(q,p)=\frac{1}{e^{-A(p-p_c)^2-B}-\sigma}=\begin{cases}\frac{1}{\exp\{-A(p-p_c)^2-B\}-1}, & \text{Bose}-\text{Einstein},\\[2ex] \frac{1}{\exp\{-A(p-p_c)^2-B\}+1}, & \text{Fermi}-\text{Dirac},\\[2ex] e^{A(p-p_c)^2+B}, & \text{Maxwell}-\text{Boltzmann}.\end{cases} \tag{3.10}$$

The physical properties of the distributions (3.10) are studied in [3] for the case of non-identical particles, and in [4] for the case of identical particles. We shall normalize these distributions as (the physical meaning of such a normalization is manifested below):

$$\int dqdp\ f_1^{(0)}(q,p)=V^{(4)}, \tag{3.11}$$

where $V^{(4)}$ is the total four-volume occupied by the ensemble in space-time. Let us introduce the system of the space-time densities, as follows:

$$\int dp\ f_1^{(0)}(q,p)\ \equiv\ n_1^{(0)}(q), \tag{3.12}$$

$$\int dp_1 dp_2 \; f_2^{(0)}(q_1,q_2,p_1,p_2) \;\; \equiv \;\; n_2^{(0)}(q_1,q_2), \tag{3.13}$$

$$\int dp_1 dp_2 dp_3 \; f_3^{(0)}(q_1,q_2,q_3,p_1,p_2,p_3) \;\; \equiv \;\; n_3^{(0)}(q_1,q_2,q_3), \quad \text{etc.} \tag{3.14}$$

Then the one-body density, $n_1^{(0)}(q)$, is normalized, in view of (3.11), as

$$\int dq \; n_1^{(0)}(q) = V^{(4)}. \tag{3.15}$$

In the case of no q-dependence of A and B, Eq. (3.13) yields $n_1^{(0)} = 1$.

4 Mean-field potential. RMS

In the equilibrium case, Eq. (2.31) for the mean-field potential entering the generalized Boltzmann equation (2.45), reduces to

$$U_1(q_1) = \frac{1}{v} \int dq_2 dp_2 \; f_1^{(0)}(q_2,p_2) V(q_1 - q_2). \tag{4.1}$$

Averaging (4.1) over the ensemble gives, through (3.11)–(3.13),

$$\begin{aligned} U \;\; &\equiv \;\; \frac{1}{2V^{(4)}} \int dq_1 dp_1 \; f_1^{(0)}(q_1,p_1) U_1(q_1) \\ &= \;\; \frac{1}{2V^{(4)} v} \int dq_1 dp_1 dq_2 dp_2 \; f_1^{(0)}(q_1,p_1) f_1^{(0)}(q_2,p_2) V(q_1 - q_2) \\ &\cong \;\; \frac{1}{2V^{(4)} v} \int dq_1 dq_2 \; n_2^{(0)}(q_1,q_2) V(q_1 - q_2), \end{aligned} \tag{4.2}$$

where we have used the relation $f_2 \approx f_1(1) f_1(2)$.

The total energy density of the ensemble is defined by

$$\rho = \rho_0 + \rho_{int}, \tag{4.3}$$

where ρ_0 is the energy density of a free gas (no-interaction case) calculated in refs. [3, 4], and ρ_{int} is the contribution of the interaction potential which is equal to

$$\rho_{int} = N_0 U,$$

N_0 being the particle number density per unit comoving "proper" three-volume $V^{(3)}$, $N_0 = N/V^{(3)}$. We now assume that for the interacting gas $V^{(4)}/N \sim v$; it then follows form (4.2) that

$$\rho = \rho_0 + \frac{N^2}{2(V^{(4)})^2 V^{(3)}} \int dq_1 dq_2 \; n_2^{(0)}(q_1,q_2) V(q_1 - q_2). \tag{4.4}$$

For a quasihomogeneous system, $n_2^{(0)}(q_1, q_2) = n_2^{(0)}(q_1 - q_2)$, so that (4.4) takes on the form

$$\rho = \rho_0 + \frac{N^2}{2V^{(4)}V^{(3)}} \int dq \, n_2^{(0)}(q) V(q). \tag{4.5}$$

By introducing hyperbolic variables for spacelike q,

$$\begin{gathered} q^0 = q \sinh\beta, \quad q^1 = q \cosh\beta \sin\theta \cos\phi, \\ q^2 = q \cosh\beta \sin\theta \sin\phi, \quad q^3 = q \cosh\beta \cos\theta, \\ 0 \le q < \infty, \ \ -\infty < \beta < \infty, \ \ 0 \le \theta \le \pi, \ \ 0 \le \phi < 2\pi, \end{gathered} \tag{4.6}$$

one can rewrite the integral in Eq. (4.5) as

$$4\pi \int q^3 dq \cosh^2\beta d\beta \, n_2^{(0)}(q) V(q).$$

This integral does not have, however, a simply interpretable nonrelativistic limit, as we discuss below after Eq. (4.12). Let us instead turn to ref. [19], where the two-body relativistic quantum-mechanical bound-state problem has been studied. It was found that, if the support of the wave function of the relative motion is restricted to an $O(2,1)$-invariant subregion of the full spacelike region, one finds a lower mass eigenvalue of the ground state than in the case when the support is in the full spacelike region. The solutions, moreover, have a simply interpretable nonrelativistic limit. This subregion was called by the authors the "restricted Minkowski space" (RMS). It has a parametrization (in contrast to (4.6) corresponding to the full spacelike region)

$$\begin{gathered} q^0 = q \sin\theta \sinh\beta, \quad q^1 = q \sin\theta \cosh\beta \cos\phi, \\ q^2 = q \sin\theta \cosh\beta \sin\phi, \quad q^3 = q \cos\theta. \end{gathered} \tag{4.7}$$

Clearly, $q_1^2 + q_2^2 - q_0^2 = q^2 \sin^2\theta \ge 0$ (and $q_1^2 + q_2^2 + q_3^2 - q_0^2 = q^2 \ge 0$ as well). This submeasure space is $O(2,1)$-invariant, but not $O(3,1)$-invariant. The representations of $O(3,1)$ are induced from the irreducible representations of $O(2,1)$ which are provided by the eigenfunctions of the two-body bound-state problem [20]. The fact that this restricted subregion admits a lower mass of the ground state than the full spacelike region constitutes a spontaneous symmetry breaking of the $O(3,1)$ invariance of the dynamical equations.

The restriction of the relative coordinates to the RMS corresponds to a restricted range of correlations available to the two events propagating in a bound state, i.e., to the range of $q_1^\mu - q_2^\mu$ available at each τ. In computing the full spectrum of the two-body problem the authors assumed that the wave functions of the excited states also lie in the $O(2,1)$-invariant subregion, i.e., these correlations are maintained for excited states as well. Indeed, it was found that the partial wave expansion for scattering theory is recovered in this submeasure space as well [21]. Here we shall assume that

this result has more generality and can be applied in statistical mechanics: *for any two events, their mutual correlations lie in the relative $O(2,1)$-invariant subregion of the full spacelike region.* It then follows that the two-body density $n_2^{(0)}(q_1-q_2)$ will have support lying in the RMS associated with the relative motion $q_1^\mu - q_2^\mu$. Therefore, the integral in Eq. (4.5) will be nonzero only in the RMS associated with q, according to the nonvanishing support of the two-body density $n_2^{(0)}(q)$. In this way we obtain for the integral in Eq. (4.5)

$$\int q^3 dq \sin^2\theta d\theta \cosh\beta d\beta d\phi \; n_2^{(0)}(q)V(q). \tag{4.8}$$

We shall also assume that, at any instant of τ, the extent of the ensemble in the q^0-direction is bounded [1], so that $V^{(4)} = V^{(3)} \cdot \triangle t$, where $\triangle t$ is the range of the time variable for the system as a whole. Therefore, in Eq. (4.8) $-\frac{\triangle t}{2} \leq q^0 \leq \frac{\triangle t}{2}$, and integration on β gives

$$\int_{-\mathrm{Arcsinh}(\triangle t/2q\sin\theta)}^{\mathrm{Arcsinh}(\triangle t/2q\sin\theta)} \cosh\beta d\beta = \frac{\triangle t}{q\sin\theta};$$

Eq. (4.8) then reduces to

$$\triangle t \int q^2 dq \sin\theta d\theta d\phi \; n_2^{(0)}(q)V(q) = 4\pi\triangle t \int dq \; q^2 n_2^{(0)}(q)V(q). \tag{4.9}$$

Using now the relations $V^{(4)} = V^{(3)} \cdot \triangle t$ and $N/V^{(3)} = N_0$, the particle number density, one finally obtains from (4.5),(4.9)

$$\rho = \rho_0 + \frac{N_0^2}{2} \int d^3q \; n_2^{(0)}(q)V(q), \tag{4.10}$$

where d^3q stands for $4\pi q^2 dq$. In the same way it is possible to obtain the expression for the pressure of the interacting gas [18]:

$$p = p_0 - \frac{N_0^2}{6} \int d^3q \; q \frac{dV(q)}{dq} n_2^{(0)}(q). \tag{4.11}$$

We see that the expressions for ρ and p are precisely of the same form as those of nonrelativistic statistical mechanics [25], but with $q \equiv \sqrt{q^\mu q_\mu}$ replacing $r \equiv \sqrt{\mathbf{q}^2}$, and $V(q)$ replacing $V(r)$. The situation is quite similar to the one occuring in the two-body bound-state problem [19], where, upon separation of variables in the RMS, one is left with a radial equation for $q \equiv \sqrt{q^\mu q_\mu}$ which is of the same form as a nonrelativistic radial Schrödonger equation for $r \equiv \sqrt{\mathbf{q}^2}$. Separation of variables in the RMS therefore has a clear correspondence to the nonrelativistic problem, as first remarked in [19]. In the nonrelativistic limit, the relative variables q^0 and p^0 vanish (all the particles are synchronized in this limit [2]), and the formulas (4.10),(4.11) acquire their standard nonrelativistic expressions [25].

We remark that the integral in Eq. (4.5) in the full spacelike region can be made convergent in the same way, by imposing the bounds on the time variable, as follows: $-\triangle t/2 \leq q \sinh\beta \leq \triangle t/2$. In this case integration on β results in the expression

$$4\pi\left(\frac{\triangle t}{2}\right)^2\int dq\; qV(q)n_2^{(0)}(q),$$

and we obtain, in place of (4.10),

$$\rho = \rho_0 + \frac{N_0^2}{2}\,\frac{\triangle t}{4}\,4\pi\int dq\; q\; n_2^{(0)}(q)V(q), \tag{4.12}$$

and similar relation for p. Hence, apart from $T_{\triangle V}$, the average passage interval in τ for the events which pass through a small representative four-volume of the system [2], contained in the expressions for ρ_0, p_0 and N_0 [3, 4], there will be another $T_{\triangle V}$ entering the expressions for ρ_{int} and p_{int}, upon replacement for $\triangle t$ in the corresponding formulas, through the relation (which represents the averaging of the equation of motion for q^0, $dq^0/d\tau = p^0/M$, over the ensemble, $\langle E\rangle$ being the average energy)

$$\triangle t = T_{\triangle V}\frac{\langle E\rangle}{M}.$$

In the nonrelativistic (or in the sharp-mass) limit, $T_{\triangle V} \rightarrow \infty$, which provides a stationarity of the system in space-time, but not a non-trivial evolution in τ [6]. While p_0, ρ_0 and N_0 are preserved in this singular limit, due to the relation [6] $T_{\triangle V}\triangle m = 2\pi$, where $\triangle m$ is the width of the mass deviation from its on-shell value, p_{int} and ρ_{int} turn out to converge with $T_{\triangle V}$. Therefore, Eq. (4.12) and similar formula for p do not have a well-defined nonrelativistic limit, in contrast to (4.10),(4.11), which admit its clear form. This fact should be a source of a spontaneous symmetry breaking of the $O(3,1)$ invariance in the correlation function of a many-body problem.

We remark that no problem with the convergence of the integral in Eq. (4.5) arises in 1+2 dimensions (for the extent in the q^0-direction bounded). Indeed, the 3D analog of (4.5) reads

$$\rho = \rho_0 + \frac{N^2}{2V^{(3)}V^{(2)}}\int d^3q\; n_2^{(0)}(q)V(q). \tag{4.13}$$

By introducing hyperbolic variables for spacelike q,

$$q^0 = q\sinh\beta, \quad q^1 = q\cosh\beta\sin\theta, \quad q^2 = q\cosh\beta\cos\theta, \tag{4.14}$$

$$0 \leq q < \infty, \;\; -\infty < \beta < \infty, \;\; 0 \leq \theta \leq 2\pi,$$

we rewrite the latter integral as

$$2\pi\int q^2 dq\cosh\beta d\beta\; n_2^{(0)}(q)V(q).$$

Integration on β gives

$$\int_{-\text{Arcsinh}(\triangle t/2q)}^{\text{Arcsinh}(\triangle t/2q)} \cosh\beta d\beta = \frac{\triangle t}{q};$$

therefore, one obtains, via $\triangle t = V^{(3)}/V^{(2)}$, $N/V^{(2)} = N_0$,

$$\rho = \rho_0 + \frac{N_0^2}{2}\int d^2q\, n_2^{(0)}(q)V(q), \tag{4.15}$$

the 3D analog of (4.10) (d^2q stands for $2\pi q dq$), and a similar relation for p.

5 Concluding remarks

We have generalized the nonrelativistic approach of Yang and Yao, based on the Wigner distribution functions and the Bogoliubov hypotheses, to the relativistic case. We have derived the generalized Boltzmann equation which, in the case of indistinguishable particles, improves the standard BUU equation in three main aspects:

1) The effect of Pauli blocking, $f \to f(1-f^{'})$, is included in the collision term. This is important for the collision processes at intermediate and low temperature, e.g., in heavy-ion collisions.

2) The modified mean-field interaction is introduced into the collision term. This has a great influence on far-nonequilibrium states.

3) The equation takes into account binary collisions corrected for many-body effects, wherein the many-body shielding effect can be obtained spontaneously.

We have introduced two-body interactions, by means of the direct action potential $V(q)$, where q is an invariant distance in the Minkowski space. The two-body correlations are taken to have the support in a relative $O(2,1)$-invariant subregion of the full spacelike region, in order to provide a good nonrelativistic limit to the basic thermodynamic quantities. Since the expressions for the energy density and the pressure are identical in form to those of the nonrelativistic theory, some of the results for the nonrelativistic interacting gas should be applicable for an interacting off-shell gas as well. For example, the equation of state of the ideal gas of non-identical particles is [3] $p = N_0T$; therefore, it follows from (4.10),(4.11) that the equation of state of a relativistic interacting gas should have the same form (by methods analogous to those of the standard cluster expansion [26]) as that of a similarly interacting nonrelativistic one, i.e. [26],

$$\frac{p}{N_0T} = \sum_{l=1}^{\infty} a_l(T)\left(\lambda^3 N_0\right)^{l-1},$$

where $\lambda \equiv \sqrt{2\pi/MT}$ is the thermal wavelength, and $a_l(T)$ is the lth virial coefficient ($a_1 = 1$).

Applications of the generalized Boltzmann equation to realistic physical systems, e.g., heavy-ion collisions, are now under consideration.

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References

[1] L.P. Horwitz, W.C. Schieve and C. Piron, Ann. Phys. **137** (1981) 306

[2] L.P. Horwitz, S. Shashoua and W.C. Schieve, Physica A **161** (1989) 300

[3] L. Burakovsky and L.P. Horwitz, Physica A **201** (1993) 666

[4] L. Burakovsky and L.P. Horwitz, Equilibrium Relativistic Mass Distribution for Indistinguishable Events, Found. Phys., *in press*

[5] L. Burakovsky and L.P. Horwitz, J. Phys. A **27** (1994) 2623

[6] L. Burakovsky and L.P. Horwitz, J. Phys. A **27** (1994) 4725

[7] L. Burakovsky and L.P. Horwitz, Relativistic Mass Distribution in Event–Anti-Event System and "Realistic" Equation of State for Hot Hadronic Matter, Found. Phys., *in press*

[8] L.P. Horwitz and C. Piron, Helv. Phys. Acta **46** (1973) 316
E.C.G. Stueckelberg, *ibid.* **14** (1941) 372, 588; **15** (1942) 23
C. Dewdney *et al.,* Phys. Lett. A **113** (1986) 359; **114** (1986) 440
A. Kyprianidis, Phys. Rep. **155** (1987) 1

[9] R.I. Arshansky, L.P. Horwitz and Y. Lavie, Found. Phys. **13** (1983) 1167

[10] J.D. Jackson, *Classical Electrodynamics,* 2nd ed., (Wiley, New York, 1975)

[11] R.I. Arshansky and L.P. Horwitz, Found. Phys. **15** (1985) 701

[12] See Dewdney *et al.* and Kyprianidis, ref. [8]. See also ref. [2]

[13] L.P. Horwitz and Y. Lavie, Phys. Rev. D **26** (1982) 819

[14] J.L. Synge, *The Relativistic Gas,* (North-Holland, Amsterdam, 1957)

[15] B.-j. Yang and S.-g. Yao, Phys. Rev. C **36** (1987) 667; Phys. Rev. A **45** (1992) 2294

[16] N.N. Bogoliubov, in *Studies in Statistical Mechanics,* ed. B. Uhlenbeck, (North-Holland, Amsterdam, 1962), Vol. I

[17] E.A. Uehling and G.E. Uhlenbeck, Phys. Rev. **43** (1933) 552

[18] L. Burakovsky and L.P. Horwitz, Manifestly Covariant Relativistic Thermodynamics and Avoidance of Gravitational Singularities, Preprint TAUP-2173-94

[19] R.I. Arshansky and L.P. Horwitz, J. Math. Phys. **30** (1989) 66

[20] R.I. Arshansky and L.P. Horwitz, J. Math. Phys. **30** (1989) 380

[21] R.I. Arshansky and L.P. Horwitz, J. Math. Phys. **30** (1989) 213

[22] W. Botermans and R. Malfliet, Phys. Lett. B **171** (1986) 22

[23] L.P. Horwitz and A. Soffer, Helv. Phys. Acta **53** (1980) 112

[24] L.P. Horwitz and F. Rotbart, Phys. Rev. D **24** (1981) 2127

[25] See R.P. Feynman, *Statistical Mechanics,* (Benjamin, New York, 1972), p. 115, for treatment of the nonrelativistic case

[26] K. Huang, *Statistical Mechanics,* (Wiley, New York, 1967), Chapter 14